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Eigenvalues of Casimir invariants for unitary irreps of $U_q(gl(m|n))$

M D Gould and J R Links

Department of Mathematics, University of Queensland, Queensland, 4072, Australia

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Abstract. A fully explicit formula for the eigenvalues of Casimir invariants for $U_q(gl(m|n))$ is given which applies to all unitary irreps. This is achieved by making some interesting observations on atypicality indices for irreps occurring in the tensor product of unitary irreps of the same type. These results have applications in the determination of link polynomials arising from unitary irreps of $U_q(gl(m|n))$.

1. Introduction

The study of $\mathbb{Z}_2$-graded or supersymmetric quantum algebras has generated substantial interest recently. Primarily this has been due to their role in solving the Yang–Baxter equation and generating associated integrable models through the quantum inverse scattering method (QISM). Since supersymmetric algebras accommodate both bosonic and fermionic degrees of freedom, such models may be interpreted as describing systems of interacting fermions and specifically correlated electrons. The approach first appeared in the work on the supersymmetric t–J model [1]. Subsequently the supersymmetric extended Hubbard model [2] and the supersymmetric U model [3] were formulated using QISM. The q-deformed analogues of the above models are discussed in [4, 5]. In fact on the open chain these models acquire quantum supersymmetry for a particular choice of boundary conditions [5]. Models derived in the context of supersymmetric realizations of the Temperley–Lieb algebra are described in [6] while the case of the Birman–Wenzl–Murakami algebra ($osp(m|2n)$ invariant) can be found in [7].

In order to gain an understanding of these models it is necessary to develop the representation theory of the underlying symmetry algebras. For the type I quantum superalgebras consisting of $U_q(gl(m|n))$ and $U_q(osp(2|2n))$ rapid and significant progress has been made. A description of the finite dimensional irreducible representations (irreps) in terms of the induced module construction has been developed [8, 9], all the unitary irreps have been classified [10, 11], the q-superdimensions of quasi-typical irreps are known [12] and the matrix elements of the $U_q(gl(m|n))$ generators in essentially typical representations have been given [13].

Recently, a formula for the eigenvalues of the Casimir invariants for the type I quantum superalgebras has been derived [14, 15] when acting on irreducible highest-weight modules. However, the formula may sometimes prove difficult to use when applied to some irreps, as seen below. The aim of this paper is to illustrate that when restricted to unitary irreps of $U_q(gl(m|n))$ these difficulties may be overcome by applying certain results on the atypicality indices for irreps occurring in the tensor product of unitary irreps of the same type. This

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approach permits us to give explicit formulae for the eigenvalues of the Casimir invariants. An important application of this result is in the evaluation of link polynomials as we will illustrate.

2. Fundamentals

Let g denote a basic classical Lie superalgebra of rank $l+1$ with usual generators $\{e_i, f_i, h_i\}_{i=0}^{l}$. Let $\{\alpha_i\}_{i=0}^{l}$ be the distinguish set of simple roots of g in the sense of Kac [16] and let $(\, , \,)$ be a fixed invariant bilinear form on H^*, the dual of the Cartan subalgebra H of g. We also let $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$ denote the full set of roots with Φ_0^+ (resp. Φ_1^+) the subset of even (resp. odd) roots. Throughout, we adopt the convention that α_0 denotes the unique odd simple root. Associated with g one can define the quantum superalgebra $U_q(g)$ which has the structure of a $\mathbb{Z}_2$-graded quasi-triangular Hopf algebra [17]. We will not give the full defining relations of $U_q(g)$ here. We note, however, that $U_q(g)$ has a co-product structure given by

$$\Delta(q^{\pm\frac{1}{2}h_i}) = q^{\pm\frac{1}{2}h_i} \otimes q^{\pm\frac{1}{2}h_i} \qquad \Delta(x) = x \otimes q^{-\frac{1}{2}h_i} + q^{\frac{1}{2}h_i} \otimes x \qquad x = e_i, f_i$$

which is extended to an algebra homomorphism to all of $U_q(g)$ in the usual way. It is important to point out that the multiplication rule for the tensor product is defined for homogeneous elements $a, b, c, d, \in U_q(g)$ by

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]}(ac \otimes bd) \tag{1}$$

and extended linearly to all of $U_q(g) \otimes U_q(g)$. Here $[a] \in \mathbb{Z}_2$ denotes the degree of the homogeneous element $a \in U_q(g)$, which is defined for the elementary generators by

$$[h_i] = 0 \qquad [e_i] = [f_i] \equiv [i] = \delta_{i0} \qquad \forall 0 \leqslant i \leqslant l$$

and extended to all homogeneous elements of $U_q(g)$ through

$$[ab] = [a] + [b] (\mathrm{mod}\, 2) \qquad \forall a, b \in U_q(g).$$

The twist map $T : U_q(g) \otimes U_q(g) \to U_q(g) \otimes U_q(g)$ is defined by

$$T(a \otimes b) = (-1)^{[a][b]} b \otimes a \tag{2}$$

for all homogeneous $a, b \in U_q(g)$: we set $\bar{\Delta} = T\Delta$. There exists a canonical element $R \in U_q(g) \otimes U_q(g)$ called the universal R-matrix which is even and invertible and satisfies the following well known relations

$$R\Delta(a) = \bar{\Delta}(a)R \qquad \forall a \in U_q(g) \tag{3}$$

$$(\Delta \otimes I)R = R_{13}R_{23} \qquad (I \otimes \Delta)R = R_{13}R_{12} \tag{4}$$

where we have adopted the conventional notation. From equations (3) and (4) it follows that the universal R-matrix satisfies the $\mathbb{Z}_2$-graded Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{5}$$

We emphasize that multiplication of the tensor products is to obey equation (1).

Let $\rho \in H^*$ denote the graded half sum of positive roots of g and let h_ρ denote the unique element of H defined by $\alpha_i(h_\rho) = (\rho, \alpha_i)$, $\forall \alpha_i \in H^*$. We recall from [18] the following result.

Theorem 1. Let π be a fixed, but arbitrary, finite dimensional representation of $U_q(g)$ with representation space V and set

$$\Delta_\pi = (\pi \otimes I)\Delta.$$

If $w \in U_q(g) \otimes \mathrm{End}\, V$ satisfies

$$\Delta_\pi(a)w = w\Delta_\pi(a) \qquad \forall a \in U_q(g) \tag{6}$$

then

$$s\tau_q(w) = (\mathrm{str} \otimes I)(\pi(q^{2h_\rho}) \otimes I)w$$

belongs to the centre of $U_q(g)$, where str denotes the supertrace. □

Theorem 1 enables a family of Casimir invariants to be constructed for $U_q(g)$ utilizing the universal R-matrix for any reference module V. Defining $R^T = TR$, it is clear from (3) that

$$R^T R\Delta(a) = \Delta(a)R^T R \qquad \forall a \in U_q(g).$$

Setting

$$A = (q - q^{-1})^{-1}(\pi \otimes I)(I \otimes I - R^T R)$$

then $A^l, l \in \mathbb{Z}^+$, satisfies (6). We thus obtain the family of Casimir invariants

$$C_l = s\tau_q(A^l). \tag{7}$$

The above discussion applies to all quantum superalgebras. Hereafter we will restrict our focus to $U_q(gl(m|n))$. For the full defining relations for $U_q(gl(m|n))$ we refer to [19].

Let $V(\mu)$ denote a finite dimensional irreducible $U_q(g)$ module of highest weight $\mu \in D^+$ where $D^+ \subset H^*$ is the set of dominant weights, and let π_μ denote the representation afforded by $V(\mu)$. For the quantum superalgebra $U_q(gl(m|n))$, $\mu \in D^+$ if and only if

$$\frac{2(\mu, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}^+ \qquad 1 \leqslant i \leqslant l$$

while (μ, α_0) can take arbitrary complex values [8]. When acting on $V(\mu)$ the invariants C_l act as scalar multiples of the identity operator (Schur's lemma), which we denote by $\chi_\mu(C_l)$. From [14, 15] we have the following result.

Theorem 2. Let λ_i denote the distinct weights in the reference module V with multiplicities m_i and $[\lambda_i]$ the degree of λ_i. The eigenvalues of the Casimir invariants (7) acting on the irreducible module $V(\mu)$ are given by

$$\chi_\mu(C_l) = \sum_i (-1)^{[\lambda_i]} m_i [\beta_i(\mu)]^l \prod_{\alpha \in \Phi_0^+} \frac{[(\mu + \lambda_i + \rho, \alpha)]_q}{[(\mu + \rho, \alpha)]_q} \prod_{\alpha \in \Phi_1^+} \frac{[(\mu + \rho, \alpha)]_q}{[(\mu + \lambda_i + \rho, \alpha)]_q} \tag{8}$$

where

$$\beta_i(\mu) = \frac{1 - q^{-(\lambda_i, \lambda_i + 2\mu + 2\rho) + (\Lambda, \Lambda + 2\rho)}}{q - q^{-1}}$$

and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

□

Recall [8] that for $U_q(gl(m|n))$ we say $\mu \in H^*$ is *typical* if

$$(\mu+\rho,\alpha) \neq 0 \qquad \forall\alpha \in \Phi_1^+$$

and *atypical* otherwise. It is apparent that the above eigenvalue formula is not well defined for those μ such that $\mu + \lambda_i$ is atypical for some λ_i. However, in principle (8) is a polynomial function of $q^{\pm(\mu,\alpha)}, \alpha \in \Phi^+$ so the right-hand expression may be expanded. This unfortunately proves to be technically difficult (e.g. see [15]). We wish to illustrate that in the case of Λ and μ being unitary of the same type, one may obtain an alternative simple expression for $\chi_\mu(C_l)$ which is always well defined.

Next we will briefly describe the unitary irreps. A more detailed discussion may be found in [10]. Define a dagger operation on the $U_q(gl(m|n))$ generators through

$$(e_i)^\dagger = f_i \qquad (f_i)^\dagger = e_i \qquad (h_i)^\dagger = h_i.$$

This operation is extended to all of $U_q(gl(m|n))$ by

$$(ab)^\dagger = b^\dagger a^\dagger \qquad \forall a, b \in U_q(gl(m|n)).$$

There are two types of unitary representations which $U_q(gl(m|n))$ admits. We say that Λ, π_Λ, $V(\Lambda)$ are type I unitary if

$$\pi_\Lambda(a^\dagger) = \pi_\Lambda(a)^\dagger$$

and type II unitary if

$$\pi_\Lambda(a^\dagger) = (-1)^{[a]}\pi_\Lambda(a)^\dagger$$

where the dagger operation for matrices denotes (non-graded) Hermitian matrix conjugation. For each unitary irrep there exists a positive definite invariant sesquilinear form which we denote $\langle\ ,\ \rangle$. These representations have the property that the tensor product of two representations of the same type reduces completely into representations also of the same type. The two types of unitary representations are in fact related by duality [10].

Throughout we choose the basis $\{\varepsilon_i\}_{i=1}^m \cup \{\delta_\nu\}_{\nu=1}^n$ for the dual of the Cartan subalgebra H^* equipped with the invariant bilinear form

$$(\varepsilon_i,\varepsilon_j) = \delta_{ij} \qquad (\delta_\mu,\delta_\nu) = -\delta_{\mu\nu} \qquad (\varepsilon_i,\delta_\mu) = 0 \qquad \forall 1 \leqslant i, j \leqslant m,\ 1 \leqslant \mu,\ \nu \leqslant n.$$

In terms of this basis we have the sets of even and odd positive roots given by

$$\Phi_0^+ = \{\varepsilon_i - \varepsilon_j | 1 \leqslant i < j \leqslant m\} \cup \{\delta_\mu - \delta_\nu | 1 \leqslant \mu < \nu \leqslant n\}$$
$$\Phi_1^+ = \{\varepsilon_i - \delta_\nu | 1 \leqslant i \leqslant m, 1 \leqslant \nu \leqslant n\}$$

and the $\mathbb{Z}_2$-graded half sum of positive roots is expressed as

$$\rho = \tfrac{1}{2}\sum_{i=1}^m (m-n-2i+1)\varepsilon_i + \tfrac{1}{2}\sum_{\mu=1}^n (m+n-2\mu+1)\delta_\mu.$$

Given an irreducible module $V(\Lambda)$, $\Lambda \in D^+$, we let $\Pi(\Lambda)$ denote the weight spectrum of $V(\Lambda)$. For later use we have the following technical results.

Lemma 1. Let $V(\Lambda)$, $\Lambda \in D^+$ be type I unitary. Then

$$(\psi,\beta) \geqslant 0 \qquad \forall\beta \in \Phi_1^+ \qquad \psi \in \Pi(\Lambda).$$

Proof. It suffices to consider the classical ($q \to 1$) case only. Let $v_\psi \neq 0$ be a weight vector of $V(\Lambda)$ with weight ψ. For $\beta = \varepsilon_i - \delta_\mu \in \Phi_1^+$ we have two possibilities; either $E^i_\mu v_\psi = 0$ or $E^i_\mu v_\psi \neq 0$ where E^i_μ denotes the standard $gl(m|n)$ generator of weight $\varepsilon_i - \delta_\mu$ under the adjoint action. In the former case we have, since $V(\Lambda)$ is type I unitary,

$$\begin{aligned} 0 &\leqslant \langle E^\mu_i v_\psi, E^\mu_i v_\psi \rangle \\ &= \langle v_\psi, E^i_\mu E^\mu_i v_\psi \rangle \\ &= (\psi, \varepsilon_i - \delta_\mu)\langle v_\psi, v_\psi \rangle \end{aligned}$$

where we have used the relation $E^i_\mu E^\mu_i + E^\mu_i E^i_\mu = E^i_i + E^\mu_\mu$. Since the form $\langle\ ,\ \rangle$ is positive definite we immediately see that $(\psi, \beta) \geqslant 0$.

In the latter case set $w = E^i_\mu v_\psi$ so that w is the weight vector of weight $\psi + \beta$. Since $(E^i_\mu)^2 = 0$ we have $E^i_\mu w = 0$ so that

$$\begin{aligned} 0 &\leqslant \langle E^\mu_i w, E^\mu_i w \rangle \\ &= \langle w, E^i_\mu E^\mu_i w \rangle \\ &= (\psi + \beta, \beta)\langle w, w \rangle. \end{aligned}$$

Thus $(\psi + \beta, \beta) = (\psi, \beta) \geqslant 0$ which is sufficient to prove the lemma. □

We have the corresponding result for type II unitary modules

Lemma 2. Let $V(\Lambda)$, $\Lambda \in D^+$ be type II unitary. Then

$$(\psi, \beta) \leqslant 0 \qquad \forall \beta \in \Phi_1^+ \qquad \psi \in \Pi(\Lambda).$$

Proof. As mentioned earlier the two types of unitary modules are related by duality [10]. Thus if $V(\Lambda)$ is type II unitary the dual module $V(\Lambda)^* = V(\Lambda^*)$ is type I unitary. The lemma then follows from the observation that the weight spectrum of $V(\Lambda^*)$ is the negative of that of $V(\Lambda)$. □

3. Atypicality indices for type I unitary irreps

Recall from [10] that $V(\Lambda)$ is a type I unitary module if and only if $\Lambda \in D^+$ is real and

$$\text{(i) } (\Lambda + \rho, \varepsilon_m - \delta_n) > 0$$

or

$$\text{(ii) } (\Lambda + \rho, \varepsilon_m - \delta_\mu) = (\Lambda, \delta_\mu - \delta_n) = 0$$

for some odd index μ. In the first instance Λ in necessarily typical whilst in the second case Λ is atypical. Let us define the set

$$E_0(\Lambda) = \{1 \leqslant i \leqslant m | (\Lambda + \rho, \varepsilon_i - \delta_{\nu_i}) = 0 \text{ for some odd index } \nu_i\}.$$

We call the integer

$$a_\Lambda = |E_0(\Lambda)|$$

the *atypicality index* of Λ. Note that in the case when Λ is typical, $E_0(\Lambda)$ is the empty set while in the case Λ is atypical we have $\nu_m = \mu$ with μ as in (ii) above. We thus see that the atypicality index of a typical weight is zero. Also the maximal possible atypicality is given by $a_0 = m \wedge n$ where $a \wedge b = \min(a, b)$.

For Λ type I unitary the set $E_0(\Lambda)$ is easily characterized. From the above we have

$$\begin{aligned} 0 &= (\Lambda + \rho, \varepsilon_i - \delta_{\nu_i}) \\ &= (\Lambda + \rho, \varepsilon_i - \varepsilon_m) + (\Lambda + \rho, \varepsilon_m - \delta_{\nu_m}) + (\Lambda + \rho, \delta_{\nu_m} - \delta_{\nu_i}) \end{aligned}$$

which in turn gives

$$(\Lambda+\rho,\delta_{\nu_m}-\delta_{\nu_i})=-(\Lambda+\rho,\varepsilon_i-\varepsilon_m)\leqslant 0.$$

Since $\Lambda+\rho$ must be dominant we deduce that $\nu_i\geqslant\nu_m$ implying $(\Lambda,\delta_{\nu_m}-\delta_{\nu_i})=0$. Thus

$$\begin{aligned}-(\Lambda+\rho,\varepsilon_i-\varepsilon_m)&=(\Lambda+\rho,\delta_{\nu_m}-\delta_{\nu_i})\\&=(\rho,\delta_{\nu_m}-\delta_{\nu_i})\\&=\nu_m-\nu_i\end{aligned}$$

or

$$\nu_i=\nu_m+(\Lambda+\rho,\varepsilon_i-\varepsilon_m). \tag{9}$$

It thus follows that for each even index i, $i\in E_0$ if and only if the odd index ν_i defined by (9) satisfies $\nu_i\leqslant n$.

Throughout we let ψ be a weight satisfying

$$\begin{aligned}&\text{(i) } (\psi,\beta)\geqslant 0 \qquad \forall\beta\in\Phi_1^+\\&\text{(ii) } \Lambda+\psi\in D^+.\end{aligned} \tag{10}$$

We aim to prove the following result on atypicality indices.

Proposition 1. If $\Lambda\in D^+$ is type I unitary and ψ satisfies (10) then

$$a_{\Lambda+\psi}\leqslant a_\Lambda.$$

□

Obviously there is nothing to prove if $\Lambda+\psi$ is typical since then $a_{\Lambda+\psi}=0\leqslant a_\Lambda$. So we assume that $\Lambda+\psi$ is atypical which also implies that Λ is atypical. Otherwise, if Λ is typical and type I unitary we would have

$$(\Lambda+\rho+\psi,\varepsilon_m-\delta_n)\geqslant(\Lambda+\rho,\varepsilon_m-\delta_n)>0$$

forcing $\Lambda+\psi$ to be typical contrary to our assumption.

For a given Λ let μ_Λ denote the odd index ν_m; i.e.

$$(\Lambda+\rho,\varepsilon_m-\delta_{\mu_\Lambda})=0.$$

Also let $i_{\Lambda+\psi}\in E_0(\Lambda+\psi)$ be the largest atypical even index and $\mu_{\Lambda+\psi}$ the corresponding odd index such that

$$\begin{aligned}&(\Lambda+\rho+\psi,\varepsilon_{i_{\Lambda+\psi}}-\delta_{\mu_{\Lambda+\psi}})=0 \qquad (\Lambda+\rho+\psi,\varepsilon_j-\delta_\nu)\neq 0\\&\forall 1\leqslant\nu\leqslant n,\ \ j>i_{\Lambda+\psi}.\end{aligned}$$

For each $i<i_{\Lambda+\psi}\in E_0(\Lambda+\psi)$ we let γ_i be the unique odd index such that

$$(\Lambda+\rho+\psi,\varepsilon_i-\delta_{\gamma_i})=0.$$

Lemma 3.

$$\begin{aligned}&\text{(i) } \mu_{\Lambda+\psi}\geqslant\mu_\Lambda\\&\text{(ii) For } i\in E_0(\Lambda+\psi) \qquad \gamma_i>\mu_{\Lambda+\psi} \qquad \forall i<i_{\Lambda+\psi}.\end{aligned}$$

Proof. (i) By definition,

$$\begin{aligned}0 &= (\Lambda+\rho+\psi, \varepsilon_{i_{\Lambda+\psi}} - \delta_{\mu_{\Lambda+\psi}})\\ &\geqslant (\Lambda+\rho, \varepsilon_{i_{\Lambda+\psi}} - \delta_{\mu_{\Lambda+\psi}})\\ &= (\Lambda+\rho, \varepsilon_{i_{\Lambda+\psi}} - \varepsilon_m) + (\Lambda+\rho, \varepsilon_m - \delta_{\mu_\Lambda}) + (\Lambda+\rho, \delta_{\mu_\Lambda} - \delta_{\mu_{\Lambda+\psi}})\\ &\Rightarrow (\Lambda+\rho, \delta_{\mu_\Lambda} - \delta_{\mu_{\Lambda+\psi}}) \leqslant -(\Lambda+\rho, \varepsilon_{i_{\Lambda+\psi}} - \varepsilon_m) \leqslant 0\end{aligned}$$

so that $\mu_\Lambda \leqslant \mu_{\Lambda+\psi}$ since $\Lambda+\rho$ is dominant.

(ii) Now for $i \in E_0(\Lambda+\psi), i < i_{\Lambda+\psi}$, we have for odd indices ν

$$\begin{aligned}(\Lambda+\rho+\psi, \varepsilon_i - \delta_\nu) &= (\Lambda+\rho+\psi, \varepsilon_i - \varepsilon_{i_{\Lambda+\psi}}) + (\Lambda+\rho+\psi, \varepsilon_{i_{\Lambda+\psi}} - \delta_{\mu_{\Lambda+\psi}})\\ &\quad+(\Lambda+\rho+\psi, \delta_{\mu_{\Lambda+\psi}} - \delta_\nu)\\ &> (\Lambda+\rho+\psi, \delta_{\mu_{\Lambda+\psi}} - \delta_\nu) \geqslant 0 \qquad \text{for } \nu \leqslant \mu_{\Lambda+\psi}.\end{aligned}$$

Thus

$$(\Lambda+\rho+\psi, \varepsilon_i - \delta_{\gamma_i}) = 0 \Rightarrow \gamma_i > \mu_{\Lambda+\psi}.$$

□

We are now in a position to prove proposition 1. It suffices to show

$$E_0(\Lambda+\psi) \subseteq E_0(\Lambda).$$

Hence suppose $i \in E_0(\Lambda+\psi)$ so that $i \leqslant i_{\Lambda+\psi}$ and

$$(\Lambda+\rho+\psi, \varepsilon_i - \delta_{\gamma_i}) = 0$$

with $\gamma_i \equiv \mu_{\Lambda+\psi}$ for $i = i_{\Lambda+\psi}$. Then

$$\begin{aligned}0 &= (\Lambda+\rho+\psi, \varepsilon_i - \delta_{\gamma_i})\\ &\geqslant (\Lambda+\rho, \varepsilon_i - \delta_{\gamma_i})\\ &= (\Lambda+\rho, \varepsilon_i - \varepsilon_m) + (\Lambda+\rho, \varepsilon_m - \delta_{\mu_\Lambda}) + (\Lambda+\rho, \delta_{\mu_\Lambda} - \delta_{\gamma_i}).\end{aligned}$$

Since from lemma 3 we have $\gamma_i \geqslant \mu_{\Lambda+\psi} \geqslant \mu_\Lambda$, then $(\Lambda, \delta_{\mu_\Lambda} - \delta_{\gamma_i}) = 0$ from which we deduce

$$\begin{aligned}0 &\geqslant (\Lambda+\rho, \varepsilon_i - \varepsilon_m) + (\rho, \delta_{\mu_\Lambda} - \delta_{\gamma_i}) = (\Lambda+\rho, \varepsilon_i - \varepsilon_m) + \mu_\Lambda - \gamma_i\\ &\Rightarrow n \geqslant \gamma_i \geqslant (\Lambda+\rho, \varepsilon_i - \varepsilon_m) + \mu_\Lambda \equiv \nu_i.\end{aligned}$$

This in turn implies that $i \in E_0(\Lambda)$; i.e.

$$i \in E_0(\Lambda+\psi) \Rightarrow i \in E_0(\Lambda)$$

which is sufficient to prove the proposition.

Given irreducible type I unitary modules $V(\Lambda)$, $V(\mu)$ consider the following direct sum decomposition into type I unitary irreducible submodules:

$$V(\Lambda) \otimes V(\mu) = \bigoplus_\nu m_\nu V(\nu)$$

with m_ν the multiplicity of the module $V(\nu)$. The weights ν are necessarily of the form

$$\nu = \mu + \psi \in D^+ \qquad \psi \in \Pi(\Lambda).$$

In view of lemma 1, ψ satisfies the conditions of (10) so that from proposition 1

$$a_\nu = a_{\mu+\psi} \leqslant (a_\mu \wedge a_\Lambda).$$

The above shows that the highest weights occurring in the tensor product of two type I unitary irreps have an atypicality index less than or equal to the atypicality index of either component. In other words the process of taking tensor products can never lead to an increase in atypicality index. Thus for a fixed integer $k \in \mathbb{Z}_+$, $k \leqslant (m \wedge n)$, we can refer to the category $\mathcal{C}_k$ of type I unitary irreps with atypicality index less than or equal to k and their direct sums. We clearly have the inclusions

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C} = \mathcal{C}_{(m \wedge n)}$$

where $\mathcal{C}_0$ is the category of typical type I unitary irreps and $\mathcal{C}$ the full category of type I unitary irreps. We have thus proved the following.

Proposition 2. The category $\mathcal{C}_k$ is closed under tensor products with arbitrary type I unitary irreps.

An interesting consequence of the above proposition is that the tensor product of a typical type I unitary irrep with an arbitrary type I unitary irrep must decompose into a direct sum of typical unitary irreps.

4. Extension to type II unitary irreps

Now we will prove that the result of proposition 1 also holds when $V(\Lambda)$, $V(\mu)$ are both type II unitary modules. Clearly the argument goes through unchanged if we can establish the result

$$a_{\mu+\psi} \leqslant a_\mu \qquad \text{for } \psi \in \Pi(\Lambda).$$

We assume that $\Lambda \in D^+$ and ψ is a weight satisfying

$$\begin{aligned} &\text{(i) } (\psi, \beta) \leqslant 0 \qquad \forall \beta \in \Phi_1^+ \\ &\text{(ii) } \Lambda + \psi \in D^+. \end{aligned} \tag{11}$$

It suffices to prove the following extension of proposition 1.

Proposition 3. If $\Lambda \in D^+$ is type II unitary and ψ satisfies (11) then

$$a_{\Lambda+\psi} \leqslant a_\Lambda.$$

□

In this case it is convenient to concentrate on the odd indices. We define the set

$$E_1(\Lambda) = \{1 \leqslant \nu \leqslant n | (\Lambda + \rho, \varepsilon_{i_\nu} - \delta_\nu) = 0 \text{ for some even index } i_\nu\}$$

so that $a_\Lambda = |E_1(\Lambda)|$. For the case of Λ being type II unitary this set may be characterized as follows. We first recall the classification scheme for type II unitary modules. For real $\Lambda \in D^+$, $V(\Lambda)$ is type II unitary if and only if

$$\begin{aligned} &\text{(i) } (\Lambda + \rho, \varepsilon_1 - \delta_1) < 0 \qquad \text{or} \\ &\text{(ii) } (\Lambda + \rho, \varepsilon_i - \delta_1) = (\Lambda, \varepsilon_1 - \varepsilon_i) = 0 \end{aligned}$$

for some even index i. In the first case Λ is necessarily typical whilst in the second case Λ is atypical.

Obviously if Λ is typical the set $E_1(\Lambda)$ is empty. If Λ is atypical and type II unitary, let i_Λ be the even index such that

$$(\Lambda + \rho, \varepsilon_{i_\Lambda} - \delta_1) = (\Lambda, \varepsilon_1 - \varepsilon_{i_\Lambda}) = 0.$$

Following the procedure employed to obtain (9), it is straightforward to show for a given odd index ν, that $\nu \in E_1(\Lambda)$ if and only if $i_\nu \geqslant 0$ where

$$i_\nu = i_\Lambda + (\Lambda + \rho, \delta_1 - \delta_\nu). \tag{12}$$

We now assume that Λ is type II unitary and ψ satisfies (11). Obviously proposition 3 holds if $\Lambda+\psi$ is typical so we assume $\Lambda+\psi$ to be atypical. We see that this implies Λ is typical otherwise

$$(\Lambda+\rho+\psi,\varepsilon_1-\delta_1)\leqslant(\Lambda+\rho,\varepsilon_1-\delta_1)<0$$

forcing $\Lambda+\psi$ to be typical, contrary to our choice of $\Lambda+\psi$. We let $\nu_{\Lambda+\psi}\in E_1(\Lambda+\psi)$ be the smallest atypical odd index and $i_{\Lambda+\psi}$ the corresponding even index so that

$$(\Lambda+\rho+\psi,\varepsilon_{i_{\Lambda+\psi}}-\delta_{\nu_{\Lambda+\psi}})=0 \qquad (\Lambda+\rho+\psi,\varepsilon_j-\delta_\nu)\neq 0$$
$$\forall 1\leqslant j\leqslant m,\ \ \nu<\nu_{\Lambda+\psi}.$$

For each $\nu\in E_1(\Lambda+\psi)$, $\nu>\nu_{\Lambda+\psi}$, let k_ν be the unique even index such that

$$(\Lambda+\rho+\psi,\varepsilon_{k_\nu}-\delta_\nu)=0.$$

We have the following analogue of lemma 3.

Lemma 4.

(i) $i_{\Lambda+\psi}\leqslant i_\Lambda$

(ii) for $\nu\in E_1(\Lambda+\psi)$ $\qquad k_\nu<i_{\Lambda+\psi}\qquad \forall\nu>\nu_{\Lambda+\psi}$.

Proof. (i) We have

$$\begin{aligned}0&=(\Lambda+\rho+\psi,\varepsilon_{i_{\Lambda+\psi}}-\delta_{\nu_{\Lambda+\psi}})\\ &\leqslant(\Lambda+\rho,\varepsilon_{i_{\Lambda+\psi}}-\delta_{\nu_{\Lambda+\psi}})\\ &=(\Lambda+\rho,\varepsilon_{i_{\Lambda+\psi}}-\varepsilon_{i_\Lambda})+(\Lambda+\rho,\varepsilon_{i_\Lambda}-\delta_1)+(\Lambda+\rho,\delta_1-\delta_{\nu_{\Lambda+\psi}})\\ &\Rightarrow(\Lambda+\rho,\varepsilon_{i_{\Lambda+\psi}}-\varepsilon_{i_\Lambda})\geqslant(\Lambda+\rho,\delta_{\nu_{\Lambda+\psi}}-\delta_1)\geqslant 0\end{aligned}$$

so that $i_{\Lambda+\psi}\leqslant i_\Lambda$ since $\Lambda+\rho$ is dominant.

(ii) For $\nu\in E_1(\Lambda+\psi)$, $\nu>\nu_{\Lambda+\psi}$, we have for indices i

$$\begin{aligned}(\Lambda+\rho+\psi,\varepsilon_i-\delta_\nu)&=(\Lambda+\rho+\psi,\varepsilon_i-\varepsilon_{i_{\Lambda+\psi}})+(\Lambda+\rho+\psi,\varepsilon_{i_{\Lambda+\psi}}-\delta_{\nu_{\Lambda+\psi}})\\ &\quad+(\Lambda+\rho+\psi,\delta_{\nu_{\Lambda+\psi}}-\delta_\nu)\\ &<(\Lambda+\rho+\psi,\varepsilon_i-\varepsilon_{i_{\Lambda+\psi}})\leqslant 0 \qquad \text{for } i\geqslant i_{\Lambda+\psi}.\end{aligned}$$

Therefore

$$(\Lambda+\rho+\psi,\varepsilon_{k_\nu}-\delta_\nu)=0\Rightarrow k_\nu<i_{\Lambda+\psi}.$$

□

To prove proposition 3, we now suppose $\nu\in E_1(\Lambda+\psi)$ so that $\nu\geqslant\nu_{\Lambda+\psi}$ and

$$(\Lambda+\rho+\psi,\varepsilon_{k_\nu}-\delta_\nu)=0$$

with $k_\nu=i_{\Lambda+\psi}$ when $\nu=\nu_{\Lambda+\psi}$. Then

$$\begin{aligned}0&=(\Lambda+\rho+\psi,\varepsilon_{k_\nu}-\delta_\nu)\\ &\leqslant(\Lambda+\rho,\varepsilon_{k_\nu}-\delta_\nu)\\ &=(\Lambda+\rho,\varepsilon_{k_\nu}-\varepsilon_{i_\Lambda})+(\Lambda+\rho,\varepsilon_{i_\Lambda}-\delta_1)+(\Lambda+\rho,\delta_1-\delta_\nu).\end{aligned}$$

Since $k_\nu\leqslant i_{\Lambda+\psi}\leqslant i_\Lambda$, we have $(\Lambda,\varepsilon_{k_\nu}-\varepsilon_{i_\Lambda})=0$ which in turn gives

$$0\leqslant i_\Lambda-k_\nu+(\Lambda+\rho,\delta_1-\delta_\nu)\Rightarrow i_\nu\equiv i_\Lambda+(\Lambda+\rho,\delta_1-\delta_\nu)\geqslant k_\nu\geqslant 0$$

indicating $\nu\in E_1(\Lambda)$ which is sufficient to prove the proposition.

The following analogue of proposition 2 is then deduced.

Proposition 4. Let $\mathcal{C}_k$ denote the category of type II unitary irreps with atypicality index less than or equal to k and their direct sums. The category $\mathcal{C}_k$ is closed under tensor products with arbitrary type II unitary irreps.

To conclude this section it is worth emphasizing that whilst propositions 1 and 3 hold for unitary irreps of the came type, a general result for atypicality indices of tensor product modules does not exist. To see this consider the tensor product $V(\Lambda) \otimes V(\Lambda)^*$ for arbitrary $\Lambda \in D^+$. The trivial (one-dimensional) module occurs in the above tensor product but has maximum atypicality index $a_0 = m \wedge n$.

5. Eigenvalue formula for unitary irreps

In this section we will use the results of propositions 1 and 3 to derive an alternative expression to (8) for the eigenvalues of the Casimir invariants (7) when acting on unitary modules $V(\mu)$. Our only assumption is that the reference representation $\pi = \pi_\Lambda$ of theorem 1 is also unitary and of the same type as π_μ.

Throughout we let $V^0(\Lambda)$ denote the $U_q(gl(m) \oplus gl(n)) \cong U_q(gl(m)) \otimes U_q(gl(n))$ module with highest weight Λ. The following result is contained in [15].

Theorem 3. Consider the $U_q(gl(m)) \otimes U_q(gl(n))$-module decomposition

$$V(\Lambda) \otimes V^0(\mu) = \bigoplus_\nu m_\nu V^0(\nu)$$

where m_ν denotes the multiplicity of the module $V^0(\nu)$ occurring in the decomposition. If μ, ν are all typical weights, we then have the $U_q(gl(m|n))$ decomposition

$$V(\Lambda) \otimes V(\mu) = \bigoplus_\nu m_\nu V(\nu).$$

The eigenvalues of the invariants C_l (with $\pi \equiv \pi_\Lambda$) on the module $V(\mu)$ are given by

$$\chi_\mu(C_l) = \sum_\nu (-1)^{[\nu]} m_\nu [\beta_\nu(\mu)]^l \prod_{\alpha \in \Phi_0^+} \frac{[(\nu+\rho,\alpha)]_q}{[(\mu+\rho,\alpha)]_q} \prod_{\alpha \in \Phi_1^+} \frac{[(\mu+\rho,\alpha)]_q}{[(\nu+\rho,\alpha)]_q}$$

where now $\beta_\nu(\mu)$ is given by

$$\beta_\nu(\mu) = \frac{1 - q^{(\Lambda,\Lambda+2\rho)+(\mu,\mu+2\rho)-(\nu,\nu+2\rho)}}{q - q^{-1}}$$

and $[\nu]$ denotes the degree of $V^0(\nu) \subset V(\Lambda) \otimes V^0(\mu)$. $\square$

Note that since the weights ν are all typical the above expression is always well defined.

For each $\mu \in D^+$, define the set

$$\Phi_1^+(\mu) = \{\alpha \in \Phi_1^+ | (\mu+\rho,\alpha) \neq 0\}$$

so that $|\Phi_1^+(\mu)| + a_\mu = |\Phi_1^+|$. We wish to prove the following.

Proposition 5. Let the reference module $V(\Lambda)$ be a unitary module. The eigenvalues of the Casimir invariants C_l, on a unitary module $V(\mu)$ of the same type, are given by

$$\chi_\mu(C_l) = \sum_{\{\nu | a_\nu = a_\mu\}} (-1)^{[\nu]} m_\nu [\beta_\nu(\mu)]^l \prod_{\alpha \in \Phi_0^+} \frac{[(\nu+\rho,\alpha)]_q}{[(\mu+\rho,\alpha)]_q} \frac{\Pi_{\alpha \in \Phi_1^+(\mu)} [(\mu+\rho,\alpha)]_q}{\Pi_{\alpha \in \Phi_1^+(\nu)} [(\nu+\rho,\alpha)]_q}.$$

Proof. Note that the above formula is well defined for all unitary $\mu \in D^+$. To see how it arises consider the one-parameter family of finite dimensional modules

$$V(\mu + \gamma\delta) \qquad \delta = \sum_{\mu=1}^{n} \delta_\mu$$

so that $(\delta, \alpha) = 1, \forall \alpha \in \Phi_1^+$. If μ is type I (resp. type II) unitary, then $\mu + \gamma\delta$ must be typical for $0 < \gamma < 1$ (resp. $0 > \gamma > -1$). For each respective case we hereafter restrict γ to this range. Consider the following decomposition with Λ, μ unitary of the same type

$$V(\Lambda) \otimes V^0(\mu + \gamma\delta) = \bigoplus m_\nu V^0(\nu + \gamma\delta).$$

Since the $\nu + \gamma\delta$ are necessarily all typical, we have from theorem 3

$$\chi_{\mu+\gamma_\delta}(C_l) = \sum_\nu (-1)^{[\nu]} m_\nu [\beta_{\nu+\gamma\delta}(\mu + \gamma\delta)]^l \\ \times \prod_{\alpha\in\Phi_0^+} \frac{[(\nu + \rho + \gamma\delta, \alpha)]_q}{[(\mu + \rho + \gamma\delta, \alpha)]_q} \prod_{\alpha\in\Phi_1^+} \frac{[(\mu + \rho + \gamma\delta, \alpha)]_q}{[(\nu + \rho + \gamma\delta, \alpha)]_q}. \tag{13}$$

Now

$$\prod_{\alpha\in\Phi_1^+} [(\mu + \rho + \gamma\delta, \alpha)]_q = \prod_{\alpha\in\Phi_1^+(\mu)} [(\mu + \rho + \gamma\delta, \alpha)]_q \prod_{\alpha\notin\Phi_1^+(\mu)} [(\gamma\delta, \alpha)]_q \\ = [\gamma]_q^{a_\mu} \prod_{\alpha\in\Phi_1^+(\mu)} [(\mu + \rho + \gamma\delta, \alpha)]_q$$

so in particular

$$\prod_{\alpha\in\Phi_1^+} \frac{[(\mu + \rho + \gamma\delta, \alpha)]_q}{[(\nu + \rho + \gamma\delta, \alpha)]_q} = [\gamma]_q^{a_\mu - a_\nu} \frac{\prod_{\alpha\in\Phi_1^+(\mu)} [(\mu + \rho + \gamma\delta)]_q}{\prod_{\alpha\in\Phi_1^+(\nu)} [(\nu + \rho + \gamma\delta, \alpha)]_q}. \tag{14}$$

Taking the limit $\gamma \to 0$, $V(\mu + \gamma\delta)$ reduces to the Kac module $K(\mu)$ [8] which has the same infinitesimal character as $V(\mu)$. This in turn implies that

$$\chi_\mu(C_l) = \lim_{\gamma\to 0} \chi_{\mu+\gamma\delta}(C_l).$$

From propositions 1 and 3, we have established that $a_\mu - a_\nu \geqslant 0$ so that (14) is well defined in the limit $\gamma \to 0$ and gives a non-zero contribution if and only if $a_\mu = a_\nu$. Thus substituting (14) into (13) and taking the limit $\gamma \to 0$ yields proposition 5. □

6. Link polynomials

The above eigenvalue formula plays an important role in the determination of link polynomials arising from unitary irreps of $U_q(gl(m|n))$. In this section we will discuss this application. We recall that the braid group B_n is generated by elements $\{b_i\}_{i=1}^{n-1}$ satisfying

$$b_i b_j = b_j b_i \qquad |i - j| > 1$$
$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}.$$

From the Yang–Baxter equation (5), representations of the braid group are obtainable in the following way. Choose a fixed homogeneous basis for $V(\Lambda)$ (assumed unitary throughout), say $\{v_\alpha\}_{\alpha=1}^d$ and let $[\alpha] \in \mathbb{Z}_2$ denote the degree of the basis vector v_α. On the tensor product module $V(\Lambda) \otimes V(\Lambda)$ we define the graded permutation operator P by

$$P(v_\alpha \otimes v_\beta) = (-1)^{[\alpha][\beta]} v_\beta \otimes v_\alpha$$

and set

$$\sigma = q^{(\Lambda,\Lambda+2\rho)} P(\pi_\Lambda \otimes \pi_\Lambda) R.$$

Then σ satisfies (cf (3) and (5))

$$[\sigma, (\pi_\Lambda \otimes \pi_\Lambda)\Delta(a)] = 0 \tag{15}$$

and

$$(I \otimes \sigma)(\sigma \otimes I)(I \otimes \sigma) = (\sigma \otimes I)(I \otimes \sigma)(\sigma \otimes I). \tag{16}$$

If we consider the nth rank tensor product space

$$V^n = V(\Lambda)^{\otimes n}$$

we obtain a $U_q(g)$ module with representation defined by

$$\pi_\Lambda^{\otimes n}(\Delta^{(n)}(a)) \qquad \forall a \in U_q(g)$$

where $\Delta^{(n)}$ is defined recursively by

$$\Delta^{(n)} = (\Delta \otimes I^{\otimes(n-2)})\Delta^{(n-1)} \qquad \Delta^{(2)} = \Delta.$$

It follows from equation (16) that the operators $\sigma_i, \sigma_i^{-1} \in \mathrm{End}(V^n)$ defined by

$$\sigma_i^{\pm 1} = I^{\otimes(i-1)} \otimes \sigma^{\pm 1} \otimes I^{\otimes(n-i-1)}$$

give rise to a representation of the braid group B_n. We have the following result from [20]:

Theorem 4. Define

$$\phi(\theta) = \frac{\mathrm{tr} \otimes \mathrm{str}^{\otimes(n-1)}[\pi_\Lambda^{\otimes n}[\Delta^{(n)}(q^{2h_\rho})\theta]]}{\mathrm{tr}\, \pi_\Lambda(q^{2h_\rho})}$$

where $\theta \in B_n$ is a word in the generators $\sigma_i^{\pm 1}$, $1 \leqslant i \leqslant n-1$, and tr and str denote the trace and supertrace respectively. Then ϕ qualifies as a Markov trace satisfying the following:

(i) $\phi(\theta\eta) = \phi(\eta\theta) \qquad \forall \theta, \eta \in B_n$

(ii) $\phi(\theta\sigma_{n-1}) = z\phi(\theta)$

$\phi(\theta\sigma_{n-1}^{-1}) = \bar{z}\phi(\theta) \qquad \forall \theta \in B_{n-1} \subset B_n$

with

$$z = \bar{z} = 1.$$

□

Let $\hat{\theta}$ denote the link obtained by closing the braid $\theta \in B_n$. A function $L(\hat{\theta})$ defines a link polynominal if it satisfies

(i) $L(\widehat{\theta\eta}) = L(\widehat{\eta\theta}) \qquad \forall \theta, \eta \in B_n$

(ii) $L(\widehat{\theta\sigma_{n-1}}) = L(\widehat{\theta\sigma_{n-1}^{-1}}) = L(\hat{\theta}) \qquad \forall \theta \in B_{n-1} \subset B_n.$

It is known that given any Markov trace $\phi(\theta)$, then $L(\hat{\theta})$ defined by

$$L(\hat{\theta}) = (z\bar{z})^{-\frac{1}{2}(n-1)} \left(\frac{\bar{z}}{z}\right)^{\frac{1}{2}e(\theta)} \phi(\theta) \qquad \theta \in B_n$$

where $e(\theta)$ is the sum of the exponents of the σ_i's appearing in θ, defines a link polynominal [21].

For the Markov trace $\phi(\theta)$ defined by theorem 1 we have

Corollary 1. $L(\hat{\theta})$ defined by

$$L(\hat{\theta}) = \phi(\theta) \qquad \theta \in B_n$$

defines a link polynominal. $\square$

In view of (15) and theorem 1 it is clear that the operators

$$c_k = (\mathrm{str} \otimes I)(\pi_\Lambda(q^{2h_\rho}) \otimes I)\sigma^k \qquad k \in \mathbb{Z}$$

are invariant operators and thus act as scalar multiples of the identity on $V(\Lambda)$. We let ζ_k denote the eigenvalue of c_k on $V(\Lambda)$. It is straightforward to show the following:

Theorem 5. For a braid $\theta \in B_n$ of the form

$$\theta = (\sigma_{i_1})^{k_1}(\sigma_{1_2})^{k_2} \ldots (\sigma_{i_{n-1}})^{k_{n-1}} \qquad k_j \in \mathbb{Z} \tag{17}$$

with $(i_1, i_2, \ldots i_{n-1})$ an arbitrary permutation of $(1, 2, \ldots, n-1)$, the corresponding link polynomial is given by

$$L(\hat{\theta}) = \prod_{i=1}^{n-1} \zeta_{k_i}.$$

$\square$

Let us now assume that the tensor product is multiplicity free; i.e.

$$V(\Lambda) \otimes V(\Lambda) = \bigoplus_\nu V(\nu).$$

In such a case powers of the braid generator are given by the expression [22]

$$\sigma^k = q^{2k(\Lambda,\Lambda+2\rho)} \sum_\nu \varepsilon_\nu q^{\frac{-k}{2}(\nu,\nu+2\rho)} P_\nu$$

where P_ν denotes the central projection onto $V(\nu)$ and ε_ν is the eigenvalue of the $\mathbb{Z}_2$-graded permutation operator P on $V(\nu)$ in the limit $q \to 1$. If we define η_ν to be the eigenvalues of the invariant operators

$$(\mathrm{str} \otimes I)(\pi_\Lambda(q^{2h_\rho}) \otimes I)P_\nu \tag{18}$$

we may write

$$\zeta_k = q^{2k(\Lambda,\Lambda+2\rho)} \sum_\nu \varepsilon_\nu q^{-k/2(\nu,\nu+2\rho)} \eta_\nu.$$

However the P_ν may be expressed [20]

$$P_\nu = \prod_{\mu \neq \nu}{}' \frac{A - \beta_\mu(\Lambda)}{\beta_\nu(\Lambda) - \beta_\mu(\Lambda)}$$

which is a polynomial in A: here the prime signifies product over the *distinct* eigenvalues of A in the set $\{\beta_\nu(\Lambda)\}$ as defined in theorem 3. Thus the invariants (18) are a polynomial function of the invariants C_l so the quantities η_ν and in turn ζ_k may be evaluated using proposition 5.

7. Example

In order to illustrate the theory we have developed we will now apply our results to the evaluation of link polynomials associated with the rank l symmetric irreps of $U_q(gl(m|n))$ which are type I unitary. Link polynomials obtained through use of the vector irreps (corresponding to the rank 1 case) give particular cases of the HOMFLY invariants as discussed in [22].

The rank l symmetric irrep has the highest weight $l\varepsilon_1$. It admits the following $U_q(gl(m) \oplus gl(n))$ decomposition

$$V(l\varepsilon_1) = \bigoplus_{k=0}^{l\wedge n} V^0((l-k)\varepsilon_1 + \delta_1 + \cdots + \delta_k)$$

from which we deduce the tensor product decomposition

$$V(l\varepsilon_1) \otimes V^0(l\varepsilon_1) = \bigoplus_{k=0}^{l\wedge n} \bigoplus_{p\leqslant l-k/2} V^0(\nu_{pk})$$

with

$$\nu_{pk} = (2l-p-k)\varepsilon_1 + p\varepsilon_2 + \delta_1 + \cdots + \delta_k.$$

For $0 < \gamma < 1$ we have the $U_q(gl(m|n))$ decomposition

$$V(l\varepsilon_1) \otimes V(l\varepsilon_1 + \gamma\delta) = \bigoplus_{k=0}^{l\wedge n} \bigoplus_{p\leqslant l-k/2} V(\nu_{pk} + \gamma\delta).$$

In [12] all the unitary irreps with maximal atypicality index were classified for $U_q(gl(m|n))$. Assuming $m > n+1$ we may deduce from these results that

$$a_{\nu_{pk}} = a_{l\varepsilon_1} \qquad \text{if and only if } k = 0.$$

Using the fact that

$$(\nu_{p0}, \nu_{p0} + 2\rho) = (2l-p)^2 + p^2 - 2p + 2l(m-n-1)$$
$$(l\varepsilon_1, l\varepsilon_1 + 2\rho) = l^2 + l(m-n-1)$$

it is then a matter of applying proposition 5 to yield

$$\chi_{l\varepsilon_1}(C_k) = \sum_{p=0}^{l} \left(\frac{1 - q^{2(p-(l-p)^2)}}{q - q^{-1}} \right)^k \frac{[2l-2p+1]_q}{[l+1]_q} \prod_{i=1}^{m-n-2} \frac{[2l-p+i+1]_q [p+i]_q}{[i]_q [l+i+1]_q}. \tag{19}$$

To determine the braid generator we observe the decomposition

$$V(l\varepsilon_1) \otimes V(l\varepsilon_1) = \bigoplus_{p=0}^{l} V(\nu_{p0})$$

from which we obtain

$$\sigma^k = q^{k(l^2 + l(m-n-1))} \sum_{p=0}^{l} (-1)^p q^{-k(2l^2 - 2lp + p^2 - p + l(m-n-1))} P_{\nu_{p0}}.$$

Using (19) and some elementary techniques of linear algebra we deduce

$$\eta_{\nu_{p0}} = \frac{[2l-2p+1]_q}{[l+1]_q} \prod_{i=1}^{m-n-2} \frac{[2l-p+i+1]_q [p+i]_q}{[i]_q [l+i+1]_q}.$$

Finally, for a braid of the form (17) we can explicitly give the associated link polynomial by the formula

$$L(\hat{\theta}) = \prod_{i=1}^{n-1} \zeta_{k_i}$$

where

$$\zeta_k = q^{k(l^2+l(m-n-1))} \sum_{p=0}^{l} (-1)^p q^{-k(2l^2-2lp+p^2-p+l(m-n-1))} \eta_{\nu_{p0}}.$$

8. Conclusion

In this paper we have derived a well defined eigenvalue formula for Casimir invariants of $U_q(gl(m|n))$ when acting on unitary modules. This formula was obtained in essence through the observation that the categories of unitary modules of fixed type with given atypicality index are closed under tensor products with arbitrary unitary modules of the same type. Our results allow the eigenvalues to be calculated in a simpler fashion than using the more general result developed previously [14, 15]. As an application of our results we have calculated new link polynomials associated with the rank l symmetric irreps for $m > n+1$.

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